LIE ISOMORPHISMS OF PRIME RINGS

BY

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1. **Introduction.** A Lie isomorphism ϕ of a ring S onto a ring R is a one-one additive mapping of S onto R which preserves commutators, i.e.,

$$\phi(x+y) = \phi(x)+\phi(y),$$
$$\phi(xy-yx) = \phi(x)\phi(y)-\phi(y)\phi(x)$$

for all $x, y \in S$. Our interest and viewpoint toward the study of Lie isomorphisms of rings was originally (and still is) inspired by the work done by I. N. Herstein on generalizing classical theorems on the Lie structure of total matrix rings to results on the Lie structure of arbitrary simple rings. In our case the starting point was the realization that it should be possible to extend the following theorem of L. Hua [1]: every Lie automorphism of the ring R of all $n\times n$ matrices over a division ring, $n>2$, characteristic $\neq 2, 3$, is of the form $\sigma+\tau$, where σ is either an automorphism or the negative of an antiautomorphism of R and τ is an additive mapping of R into its center which maps commutators into zero. Indeed, using some of Hua's techniques and some valuable suggestions due to Nathan Jacobson, we were in [2], roughly speaking, able to obtain the same conclusion under the weaker assumption that R was merely a primitive ring possessing three orthogonal idempotents whose sum was 1. In a recent paper [3], while making the stronger assumption that R was simple, we were able to lower the number of idempotents from three to two. For the most part, the same techniques were used in this second paper, although a tensor product method due to Jacobson was to replace tedious calculations involving matrix units due to Hua, and some results of Herstein on Lie ideals of simple rings seemed necessary.

Our goal in this paper is Theorem 11, in which we extend the above results to the situation where R is a prime ring with two orthogonal idempotents whose sum is 1. Primeness is a natural generalization of simplicity and primitivity, and, in the sense of keeping free of the radical and of (sub)direct sums of ideals, it is perhaps the strongest generalization one may make. Whether the assumption of idempotents is necessary or not is still a major open question. In all our work on the subject (including the present paper) our arguments rest heavily on the presence of a nontrivial idempotent. A successful removal of the assumption of idempotents would certainly require totally new methods; one would, for example, have to face the situation of an arbitrary division ring.

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A more accurate statement of our result for Lie isomorphisms of primitive rings would reveal the fact that the image of R under the (anti)isomorphism σ (and hence under τ) may in general be contained in a larger ring than R itself. Indeed, an example given in [2, p. 916], shows that the image of σ need not be contained in R. Fortunately, in this case we could consider R as a "dense" subring of the ring of all linear transformations of a vector space over a division ring. So, although we still could (and did) use most of the techniques as in [3] for the case where R is a prime ring, we were confronted with the new problem of finding a suitable extension ring of R which would contain the images of σ and τ.

A choice for an extension ring of R which proves successful for the prime ring case is the so-called complete ring of right quotients Q of R. A good account of the construction and properties of this ring is given by Lambek in [4, Chapter 4]. In §2 of this paper we give our version (basically Utumi's formulation of Q) of how we feel Q should be characterized. The remainder of the section is devoted to the problem of, given a prime ring R with complete ring of right quotients Q, and given several rings closely related to R, the determination of the complete ring of right quotients of these related rings in terms of Q. The results we obtain, Theorems 1, 2, and 3, are very straightforward, and, although we have not explicitly seen them stated in the literature, we believe that they are all more or less well known. For completeness, however, we provide the details of the proofs.

In §3 a key result (Theorem 4) and useful corollary (Theorem 5) are proved. Theorem 4 says that in a prime ring R, if $axb=bxa$ for all $x \in R$, then a and b are C-dependent, where C is the center of the complete ring of right quotients Q. This generalizes to prime rings a result of Amitsur [5, p. 215, Lemma 6(a)] for primitive rings. Theorem 5, a theorem on tensor products, has as an important application (pointed out first to the author by Jacobson), Theorem 6, which describes idempotents in terms of the Lie product.

The study of Lie isomorphisms proper is begun in §4. Theorem 7, which describes what happens to idempotents under Lie isomorphisms, splits our problem into what we call Case 1 and Case 2. Theorem 8, in which Herstein's Lie theory of simple rings appears, reduces each case into two further cases (Cases 1a, 1b, 2a, 2b). The remainder of this section is devoted to the proof of Theorem 9, the main theorem for Case 1a.

In §5, the main theorem for Case 1b, Theorem 10, is first worked out. A device (first mentioned to us by Jacobson) reduces Case 2 to Case 1, and so we quickly reach Theorem 11, the main theorem of our paper. Theorems 12, 13, and 14 then show that previous results of ours on Lie isomorphisms of simple and primitive rings follow as corollaries of Theorem 11.

2. **The complete ring of quotients of a prime ring.** The notion of a complete ring of right quotients of an arbitrary ring R has been given by Lambek and Findley [6], Utumi [7], and others. We choose to use the formulation due to

Utumi and proceed to review some of the relevant definitions. A right ideal D of a ring R is said to be *dense* if, for any given pair of ring elements r_1, r_2 with $r_1 \neq 0$, there exists an $r \in R$ such that $r_1 r \neq 0$ and $r_2 r \in D$. If D and D' are dense right ideals of R, J is a right ideal of R containing D, and $a \in R$, then the sets J, $D \cap D'$, and $a^{-1} = \{x \in R \mid ax \in D\}$ are easily seen to be dense right ideals. Let $W(R) = \{x \in R \mid xD = 0$ for some dense right ideal $D\}$. One shows, using the preceding remark, that $W(R)$ is a two-sided ideal of R. The relevance of $W(R)$ will be evident after we now give the characterizing properties of the complete ring of right quotients Q $(= Q(R))$ of a ring R.

Definition. A ring Q is said to be a complete ring of right quotients of a ring R if:

(1) $R \subseteq Q$.

(2) If $f \in \mathrm{Hom}_R(D, R)$, where D is a dense right ideal of R, then there exists $q \in Q$ such that $qd = f(d)$ for all $d \in D$.

(3) If $q \in Q$, there is a dense right ideal D of R such that $qD \subseteq R$.

(4) For all $q \in Q$, $q = 0$ if and only if $qD = 0$ for some dense right ideal D of R.

We now state (and outline part of the proof of) the following remark: Q exists if and only if $W(R) = 0$ and is unique up to isomorphism.

For a construction of Q one proceeds by first forming the set

$$P = \bigcup \{\mathrm{Hom}_R(D, R) \mid D \text{ dense right ideal of } R\}$$

and defining elements f (acting on D) and g (acting on D') of P to be equivalent if $f = g$ on $D \cap D'$. Let Q then be the set of equivalence classes. Addition and multiplication are basically given by the usual addition and composition of representatives acting on suitably restricted domains. We show first that R may be isomorphically embedded in Q (and then, for simplicity, identify R with its isomorphic image). Indeed, consider the mapping $a \to \hat{a}_l$ of R into Q, where $\hat{a}_l$ is the equivalence class determined by the left multiplication a_l acting on R. Then $\hat{a}_l = 0$ means that $aD = 0$ for some dense right ideal D of R, and so $a = 0$ since $W(R) = 0$. The proofs of properties (2), (3), and (4) are straightforward, with $W(R) = 0$ again being needed to show (4). Conversely, property (4) certainly implies that $W(R) = 0$.

Next suppose that $x_1 \to x_2$ is an isomorphism of R_1 onto R_2, and suppose that Q_1 and Q_2 are complete rings of right quotients of R_1 and R_2, respectively. If $\hat{f}_1 \in Q_1$, with $f_1 : x_1 \to y_1$ an R_1-homomorphism of D_1 into R_1, then $f_2 : x_2 \to y_2$ is easily seen to be an R_2-homomorphism of the dense right ideal D_2 into R_2. Using the fact that $W(R) = 0$, one shows that $\hat{f}_1 \to \hat{f}_2$ gives the desired ring isomorphism of Q_1 onto Q_2.

The condition $W(R) = 0$ is satisfied in particular whenever the so-called right singular ideal $Z(R) = 0$, since every dense right ideal is essential. We are also assured that $W(R) = 0$ if $1 \in R$. In fact, suppose $aD = 0$ for some dense right ideal

D of R, where $a \neq 0$. Then there exists $r \in R$ such that $ar \neq 0$ but $1 \cdot r = r \in D$, a contradiction.

We now develop several results which indicate the connection between the complete ring of quotients of a ring R and the complete ring of quotients of a ring related to R. We do not assume that $1 \in R$. Although these results are undoubtedly known, we have not seen explicit statements of them and therefore will provide proofs.

LEMMA 1. *Let $0 \neq U$ be an ideal of a prime ring R, and let $f \in \operatorname{Hom}_U(D, U)$, where D is a dense right ideal of U. Then DU is a dense right ideal of R and $f \in \operatorname{Hom}_R(DU, R)$.*

Proof. Let $r_1 \neq 0$, $r_2 \in R$. Since R is prime $r_1 u \neq 0$ for some $u \in U$. Because D is dense in U, there exists $v \in U$ such that $(r_1 u)v \neq 0$ and $(r_2 u)v \in D$. Again because R is prime there exists $w \in U$ such that $r_1(uvw) \neq 0$. Since $r_2 uvw \in DU$, we have shown that DU is a dense right ideal of R. For $d \in D$, $u \in U$, $r \in R$ note that

$$f((du)r) = f(d(ur)) = f(d)(ur) = f(du)r,$$

i.e., $f \in \operatorname{Hom}_R(DU, R)$.

COROLLARY. *U is a dense right ideal of R.*

THEOREM 1. *Let $0 \neq U$ be an ideal of a prime ring R. Then $Q(U) = Q(R)$.*

Proof. Let $f \in \operatorname{Hom}_U(D, U)$, where D is a dense right ideal of U. By Lemma 1, DU is a dense right ideal of R and $f \in \operatorname{Hom}_R(DU, R)$. Then there exists $q \in Q(R)$ such that $qdu = f(du)$ for all $d \in D$ and $u \in U$. But $f(du) = f(d)u$, and so $(qd - f(d))U = 0$, which forces $qd = f(d)$, by property (4). This completes the proof of (2). Next, if $q \in Q(R)$, there is a dense right ideal D of R such that $qD \subseteq R$. If $u_1 \neq 0$, $u_2 \in U$, there exists $r \in R$ such that $u_1 r \neq 0$ and $u_2 r \in D$. Because R is prime we can find $u \in U$ such that $u_1(ru) \neq 0$. As $u_2(ru) \in DU$, we see that DU is a dense right ideal of U. $q(DU) \subseteq RU \subseteq U$, and so (3) is proved. Finally, suppose $qD = 0$ for some dense right ideal D of U. By Lemma 1, DU is a dense right ideal of R. Then $q(DU) = 0$ and thus $q = 0$, showing (4).

If $e \neq 0, 1$ is an idempotent in a ring R, we set $e_1 = e$ and write $R_{11} = e_1 R e_1$, $R_{12} = e_1 R(1 - e_1)$, $R_{21} = (1 - e_1)R e_1$, and $R_{22} = (1 - e_1)R(1 - e_1)$. Then R may be written in its Peirce decomposition $R_{11} \oplus R_{12} \oplus R_{21} \oplus R_{22}$. Note that R need not have an identity element.

LEMMA 2. *Let R be a prime ring containing an idempotent $e \neq 0, 1$. Then*

(a) *If D is a dense right ideal of eRe, $DR + (1 - e)R$ is a dense right ideal of R.*

(b) *If D is a dense right ideal of R, $D \cap eRe$ is a dense right ideal of eRe.*

Proof. Let $r \neq 0$, $s \in R$, and write $s = s_{11} + s_{12} + s_{22} + s_{21}$, $s_{ij} \in R_{ij}$. We can find $x, y \in R$ such that $a_1 = e_1 x r y e_1 \neq 0$, since R is prime. Now write $sye_1 = t_{11} + t_{12} + t_{21} + t_{22}$. Since D is a dense right ideal of R_{11}, there exists $u_1 \in R_{11}$ such that $a_1 u_1 \neq 0$

and $t_{11}u_1 \in D$. From $a_1u_1 \neq 0$ we have $r(ye_1u_1) \neq 0$ and $s(ye_1u_1) = t_{11}u_1 + t_{21}u_1 \in D + (1-e)R$. Thus (a) is proved. To verify (b) let $x_1 \neq 0$, $y_1 \in R_{11}$. There exists $r \in R$ such that $x_1r \neq 0$ and $y_1r \in D$. There exists $c \in R$ such that $x_1rce_1 \neq 0$, since R is prime. Then $x_1(e_1rce_1) = x_1rce \neq 0$, and $y_1(e_1rce_1) = (y_1r)ce_1 \in D \cap R_{11}$.

LEMMA 3. *Let $f \in \mathrm{Hom}_{R_{11}}(D, R_{11})$, where D is a dense right ideal of R_{11}. Then f can be extended to $g \in \mathrm{Hom}_R(E, R)$, where $E = DR + (1-e)R$.*

Proof. Noting that $DR = D + DR_{12}$, we may write an element $x \in E$ in the form $x = d + \sum_\lambda d^\lambda x_{12}^\lambda + y$, $d, d^\lambda \in D$, $x_{12}^\lambda \in R_{12}$, $y \in (1-e)R$. We then define a mapping $g: E \to R$ by setting $g(x) = f(d) + \sum_\lambda f(d^\lambda)x_{12}^\lambda$. To see that g is well defined, suppose $d + \sum_\lambda d^\lambda x_{12}^\lambda + y = 0$. Thus $d = 0$, $\sum_\lambda d^\lambda x_{12}^\lambda = 0$, and $y = 0$. We choose $y_{21} \in R_{21}$ and note that $\sum_\lambda d^\lambda(x_{12}^\lambda y_{21}) = 0$. Applying f, we have

$$0 = \sum_\lambda f(d^\lambda)(x_{12}^\lambda y_{21}) = \left\{\sum_\lambda f(d^\lambda)x_{12}^\lambda\right\} y_{21} = 0.$$

In other words, $\{\sum_\lambda f(d^\lambda)x_{12}^\lambda\}Re_1 = 0$, and so, by the primeness of R, $\sum_\lambda f(d^\lambda)x_{12}^\lambda = 0$. g is thus a well-defined mapping, and clearly it is additive and extends f. Now let $r \in R$ and write $r = r_{11} + r_{12} + r_{21} + r_{22}$. For $d \in D$,

$$\begin{aligned} g(dr) &= g(dr_{11} + dr_{12}) = f(dr_{11}) + f(d)r_{12} = f(d)r_{11} + f(d)r_{12} \\ &= f(d)(r_{11} + r_{12}) = g(d)r. \end{aligned}$$

Furthermore,

$$\begin{aligned} d(dx_{12}r) &= g\{dx_{12}(r_{22} + r_{21})\} = d\{d(x_{12}r_{22}) + d(x_{12}r_{21})\} = f(d)x_{12}r_{22} + f(d)x_{12}r_{21} \\ &= g(dx_{12})(r_{22} + r_{21}) = g(dx_{12})r. \end{aligned}$$

It follows that $g \in \mathrm{Hom}_R(E, R)$.

THEOREM 2. *Let R be a prime ring containing an idempotent $e \neq 0, 1$. Then*

$$Q(eRe) = eQ(R)e.$$

Proof. We set $Q = Q(R)$ and note first that $eRe \subseteq eQe$. Next let $f \in \mathrm{Hom}_{R_{11}}(D, R_{11})$, D a dense right ideal of R_{11}. By Lemma 3, f may be extended to $g \in \mathrm{Hom}_R(E, R)$, where $E = DR + (1-e)R$ is, by Lemma 2(a), a dense right ideal of R. Therefore there exists $q \in Q$ such that $qv = g(v)$ for all $v \in E$. In particular, $qd = f(d)$ for all $d \in D$. Hence $eqed = f(d)$ and (2) is proved. Now let $q \in eQe \subseteq Q$. There is a dense right ideal D of R such that $qD \subseteq R$. By Lemma 2(b), $E = D \cap eRe$ is a dense right ideal of eRe. Since $qE \subseteq R \cap eRe = eRe$, (3) has been shown. Finally, suppose $q \in eQe$ such that $qD = 0$, where D is a dense right ideal of eRe. By Lemma 2(a), $E = DR + (1-e)R$ is a dense right ideal of R. Then $qE = 0$, which forces $q = 0$.

LEMMA 4. *If T is a ring contained between R and $Q(R)$ and D is a dense right ideal of T, then $D \cap R$ is a dense right ideal of R.*

Proof. Let $r_1 \neq 0$, $r_2 \in R$ and find $t \in T$ such that $r_1t \neq 0$ and $r_2t \in D$. From (3) and (4) of the definition of $Q(R)$, there exists a dense right ideal E of R such that

$0 \neq tE \subseteq R$, and hence that $0 \neq (r_1t)E \subseteq R$. Therefore there is an $x \in E$ such that $0 \neq r_1(tx)$ and $r_2(tx) \in D \cap R$.

THEOREM 3. *Let T be a ring contained between R and $Q(R)$. Then $Q(T) = Q(R)$.*

Proof. Let $f \in \mathrm{Hom}_T(D, T)$, D a dense right ideal of T. By Lemma 4, $D \cap R$ is a dense right ideal of R. Let $E = \{x \in D \cap R \mid f(x) \in R\}$. To see that E is a dense right ideal of R, let $r_1 \neq 0$, $r_2 \in R$ and pick $r \in R$ such that $r_1r \neq 0$ and $r_2r \in D \cap R$. Since $f(r_2r) \in Q$, there is a dense right ideal F of R such that $f(r_2r)F \subseteq R$. Since $r_1r \neq 0$, $r_1rF \neq 0$. Hence there is an $x \in F$ such that $r_1(rx) \neq 0$. Also $f(r_2rx) = f(r_2r)x \in R$, and so $r_2(rx) \in E$. It follows that $f \in \mathrm{Hom}_R(E, R)$. Thus there is $q \in Q$ such that $qv = f(v)$ for all $v \in E$. Now let $d \in D$. We know that $G = \{x \in R \mid dx \in E\}$ is a dense right ideal of R. In view of this $\{qd - f(d)\}x = q(dx) - f(dx) = 0$ for all $x \in G$. Therefore $qd = f(d)$ for all $d \in D$ and (2) has been proved.

Next let $q \in Q$. $qD \subseteq R$ for some dense right ideal of R. Let $t_1 \neq 0$, $t_2 \in T$, and choose $x \in R$ so that $0 \neq t_1x \in R$ and $t_2x \in R$. There is an $r \in R$ such that $t_1(xr) \neq 0$ and $t_2(xr) \in D$. Hence DT is a dense right ideal of T and $q(DT) \subseteq RT \subseteq T$, showing (3). Finally, suppose $qD = 0$ for some dense right ideal D of T. By Lemma 4, $D \cap R$ is a dense right ideal of R. Since $q(D \cap R) = 0$, we obtain $q = 0$. This completes the proof that $Q(T) = Q(R)$.

We conclude this section by remarking that if R is a prime ring, then the three related rings studied in Theorems 1, 2, and 3 are also prime rings. Indeed, it is well known that a nonzero ideal U of R is again a prime ring, and that eRe is a prime ring. To see that T is a prime ring, where $R \subseteq T \subseteq Q$, suppose $aTb = 0$, with $a \neq 0$, $b \neq 0 \in T$. As we have seen before, there is a dense right ideal D of R such that $0 \neq aD \subseteq R$ and $0 \neq bD \subseteq R$. Choose $x, y \in D$ such that $ax \neq 0$ and $by \neq 0$. Then $(ax)R(by) \subseteq aTby = 0$, a contradiction.

3. **Extending the center of a prime ring.** We assume throughout this section that R is a prime ring such that $W(R) = 0$, i.e., such that $Q = Q(R)$, the complete ring of right quotients of R, exists. C will denote the center of Q. We always have $1 \in C$, although R does not necessarily have an identity element. $R_C = C + RC$ will denote the subring of Q generated by R and C.

LEMMA 5. *$C = \{q \in Q \mid qx = xq$ for all $x \in R\}$ and C is a field.*

Proof. Suppose $qx = xq$ for all $x \in R$. Let $w \in Q$, and choose a dense right ideal D of R such that $wD \subseteq R$. For $d \in D$, $(qw)d = q(wd) = (wd)q = w(dq) = q(qd) = (wq)d$. It follows that $qw = wq$, and so $q \in C$.

Next let $0 \neq c \in C$. There exists a dense right ideal D of R such that $0 \neq cD \subseteq R$. Let $r_1 \neq 0$, $r_2 \in R$. There is an $x \in D$ such that $r_1x \neq 0$ since $r_1 \neq 0$. We next find a $y \in R$ such that $r_1xy \neq 0$ and $r_2xy \in D$. Then $r_1(cxy) = (r_1xy)c \neq 0$ (since c is a regular element) and $r_2(cxy) = (r_2xy)c \in Dc = cD$. We have thus shown that cD is a dense right ideal of R. The mapping $f: cD \to R$ given by $f(cd) = d$, $d \in D$, is a well-defined R-homomorphism since c is regular. There is a $q \in Q$ such that $qcd = f(cd) = d$ for all $d \in D$. Therefore $qc = 1$, and so C is a field.

COROLLARY. *Let R be a prime ring with $W(R) = 0$. Then R_C is a prime ring whose center is the field C.*

THEOREM 4. *Let R be a prime ring with $W(R) = 0$, and suppose $axb = bxa$ for some $a, b \in R$ and for all $x \in R$. Then a and b are C-dependent.*

Proof. We may assume that $a \neq 0$ and $b \neq 0$. Let $D = RaR$. By the corollary of Lemma 1, D is a dense (right) ideal of R. We define a mapping $f: D \to R$ according to the rule

$$\sum_i x_i a y_i \to \sum x_i b y_i, \qquad x_i, y_i \in R.$$

To show that f is well defined, we suppose that $\sum x_i a y_i = 0$. Then $0 = br \sum x_i a y_i = \sum b(rx_i)ay_i$ for all $r \in R$, and thus, by our hypothesis, $\sum a(rx_i)by_i = ar(\sum x_i b y_i) = 0$. Since R is prime we conclude that $\sum x_i b y_i = 0$, showing that f is well defined. f is an R-homomorphism because

$$f\{(xay)r\} = f\{xa(yr)\} = xb(yr) = f(xay)r$$

for all $x, y, r \in R$. Now, from (2) of the definition of Q, we may find $q \in Q$ such that $qd = f(d)$ for all $d \in D$. If $r \in R$, then $qr(xay) = q(rxay) = f(rxay) = rxby = rf(xay) = rq(xay)$, showing that $(qr - rq)D = 0$. Therefore $qr = rq$ for all $r \in R$, and so, by Lemma 5, $q \in C$. In particular, $x(qa)y = q(xay) = f(xay) = xby$ for all $x, y \in R$, i.e., $R(qa - b)R = 0$. Since R is prime, we obtain $qa = b$.

We pause at this point to mention an example (communicated orally to us by P. M. Cohn) which shows that C may properly contain Z, the center of R, even if Z is a field. In fact, let F be a field possessing an automorphism σ of infinite period. Let R be the "skew" polynomial ring $F[x, y]$, with the usual addition and with multiplication conforming to the rules $xy = yx$, $xa = a^\sigma x$, and $ya = a^\sigma y$, for all $a \in F$. R is a right and left Noetherian integral domain whose center

$$Z = \{a \in F \mid a^\sigma = a\}.$$

One can then show that (in $Q(R)$) $xy^{-1} \in C$ but $xy^{-1} \notin R$.

THEOREM 5. *Let R be a prime ring with $W(R) = 0$, and let $T = R_C$. Then $T' \otimes_C T \cong T_l T_r$, where T' is the opposite ring of T and T_l (T_r) is the ring of left (right) multiplications of T acting on T.*

Proof. One checks in a straightforward way that the mapping

$$\sum_i a'_i \otimes b_i \to \sum_i a_{il} b_{ir}, \qquad a'_i \in T', \quad b_i \in T,$$

is a homomorphism of the ring $T' \otimes_C T$ onto the ring $T_l T_r$. Suppose the kernel $K \neq 0$. Choose a nonzero element $\sum_{i=1}^m a'_i \otimes b_i$ in K with minimal "length" m.

Then $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$ are both independent sets over C. For $x \in T$

$$(a'_m x' \otimes 1)\left(\sum_{i=1}^{m} a'_i \otimes b_i\right) - \left(\sum_{i=1}^{m} a'_i \otimes b_i\right)(x'a'_m \otimes 1) = \sum_{i=1}^{m-1} (a'_m x' a'_i - a'_i x' a'_m) \otimes b_i$$

is an element in K of length less than m, and so must be 0. By the independence of $\{b_i\}$, one sees that

$$a_m x a_i - a_i x a_m = 0, \qquad i = 1, 2, \ldots, m,$$

for all $x \in T$. By Theorem 3, $Q(T) = Q(R)$, and so, by applying Theorem 4 to the ring T, we have $a_m = c_i a_i$, $c_i \in C$, $i = 1, 2, \ldots, m-1$. Thus a contradiction is reached, since the $\{a_i\}$ are C-independent.

COROLLARY. *Let R be a prime ring with* 1, *and let $T = RC$. Suppose $\sum_{i=1}^{n} a_i x b_i = 0$ for all $x \in T$. Then either $\{a_i\}$ are C-dependent or $\{b_i\}$ are C-dependent.*

Proof. Suppose $\{a_i\}$, $\{b_i\}$ are both C-independent sets. Since $\sum_i a_{il} b_{ir} = 0$, by the theorem $\sum a'_i \otimes b_i = 0$, a contradiction.

We only mention this corollary because we feel it may shed some light on the problem of attempting to generalize Amitsur's results on generalized polynomial identities for primitive rings [5] to analogous results for prime rings. Our corollary amounts in effect to a statement that there does not exist a prime ring R with 1 which satisfies a nontrivial "generalized" polynomial identity of degree 1.

For the remainder of this section we apply Theorem 5 to obtain some needed results on the structure of R as a Lie ring.

THEOREM 6. *Let R be a prime ring with $W(R) = 0$, of characteristic $\neq 2, 3$, and let a be an element of $T = R_C$. Then $a = e + c$, e an idempotent in T, $c \in C$, if and only if*

$$(1) \qquad [[[xa]a]a] = [xa] \quad \textit{for all } x \in R.$$

(*Note that here and frequently throughout the rest of the paper* $[xa]$ *will stand for* $xa - ax$.)

Proof. If $a = e + c$, (1) is straightforward to verify. Conversely, if (1) holds, it is easy to first check that

$$(2) \qquad [[[xa]a]a] = [xa] \quad \text{for all } x \in T.$$

(2) may be written as

$$(3) \qquad a_r^3 - 3a_l a_r^2 + 3a_r a_l^2 - a_l^3 = a_r - a_l.$$

Since $T_l T_r \cong T' \otimes_C T$ by Theorem 5, we may replace (3) by

$$(4) \qquad 1 \otimes a^3 - 3a \otimes a^2 + (3a^2 - 1) \otimes a + (a - a^3) \otimes 1 = 0.$$

Therefore $\{1, a, a^2, a^3\}$ is a C-dependent set. If $a \in C$, we are already finished, so we may assume that $\{1, a\}$ is an independent set. If $\{1, a, a^2\}$ is an independent set, then $a^3 = \alpha a^2 + \beta a + \gamma$, $\alpha, \beta, \gamma \in C$. Substituting this in (4) yields

$$1 \otimes \{\alpha a^2 + (\beta - 1)a\} + a \otimes \{-3a^2 + (1-\beta)a\} + a^2 \otimes (3a - \alpha) = 0.$$

In particular, $3a = \alpha$, a contradiction since the characteristic of T is different from 3. We are thus forced to assume

$$a^2 = \alpha a + \beta, \tag{5}$$

whence

$$a^3 = (\alpha^2 + \beta)a + \alpha\beta. \tag{6}$$

Substitution of (5) and (6) in (4) yields

$$1 \otimes (\alpha^\rho + 4\beta - 1)a + a \otimes (1 - \alpha^2 - 4\beta) = 0$$

or

$$\alpha^2 + 4\beta - 1 = 0. \tag{7}$$

Using (5) and (7) one verifies directly that $e = a + \frac{1}{2}(1-\alpha)$ is an idempotent, and thus $a = e + c$, where $c = -\frac{1}{2}(1-\alpha)$.

The next three lemmas are due to Herstein.

Lemma 6. *Let R be a prime ring of characteristic $\neq 2$, let I be a nonzero ideal of R, and suppose $[[xa]a] = 0$ for all $x \in I$. Then a lies in the center of R.*

Proof (Herstein). From $[[xy, a]a] = 0$ and $[xy, a] = x[y, a] + [x, a]y$, $x, y \in I$, and char $R \neq 2$, one obtains $[x, a][y, a] = 0$. Setting $y = rx$, $x, r \in I$, one sees that $[x, a]r[x, a] = 0$, i.e., $[x, a]I[x, a] = 0$. By the primeness of R, $[x, a] = 0$ for all $x \in I$. Replacing $x \in I$ by $xr \in I$, $r \in R$, we then have $x(ra - ar) = 0$ for all $x \in I$, which forces $ra - ar = 0$ for all $r \in R$.

Lemma 7. *Let R be a prime ring such that $[[R, R], [R, R]] = 0$. Then R is commutative.*

Proof. If $a \notin Z$ (the center of R), then by Lemma 6 there is an $x \in R$ such that $b = [x, a] \notin Z$. But by hypothesis $[[y, b]b] = 0$ for all $y \in R$, a contradiction to Lemma 6.

Lemma 8. *Let R be a prime ring of characteristic $\neq 2$, and let U be a Lie ideal of R. Then either $U \subseteq Z$ or $U \supseteq [I, R]$ for some nonzero ideal I of R.*

Proof. Let $T(U) = \{x \in R \mid [x, R] \subseteq U\}$. $T(U)$ is easily seen to be a Lie ideal of R. Furthermore, letting $a, b \in T(U)$, $r \in R$, and writing

$$(ab)r - r(ab) = a(br) - (br)a + b(ra) - (ra)b,$$

one sees that $T(U)$ is also an associative subring of R. By [8, p. 9, Lemma 2], either $T(U) \subseteq Z$ or $T(U) \supseteq I$ for some ideal I of R. If $T(U) \subseteq Z$, then $U \subseteq T(U) \subseteq Z$. If $T(U) \supseteq I$, then $U \supseteq [T(U), R] \supseteq [I, R]$.

LEMMA 9. *Let R be a prime ring with* 1 *and let a and b be idempotents of $T=R_C(=RC)$ such that $ab=ba$ and*

$$[[[[xa]b]a]b]+[[xa]b] = 0 \quad \textit{for all } x \in R. \tag{1}$$

Then either $ab=0$ or $(1-a)(1-b)=0$.

Proof. (1) holds in the prime ring T. Multiplication of (1) on the left by ab then gives us

$$abxab-abxb-abxa+abx = 0, \qquad x \in T.$$

Factoring, we have $abT(1-a)(1-b)=0$, from which one concludes that either $ab=0$ or $(1-a)(1-b)=0$.

LEMMA 10. *Let R be a prime ring with $W(R)=0$, and let $e\neq 0$, 1 be an idempotent in $T=T_C$. Then the center of eTe is eC.*

Proof. Suppose $(eae)(exe)=(exe)(eae)$ for all $x \in R$. Setting $b=eae$, we have $bxe=exb$ for all $x \in R$, and indeed for all $x \in T$. We then apply Theorem 4 directly to the ring T to conclude that $b=ce$, for some $c \in C$.

4. **Lie isomorphisms of prime rings: Case 1a.** Henceforth in this paper we shall assume that S is a prime ring with 1 of characteristic different from 2 and 3, and containing two nonzero orthogonal idempotents e_1 and e_2 whose sum is 1. We assume further that there exists a Lie isomorphism ϕ of S onto a prime ring R with 1. We remark that $\phi(x) \in Z$, the center of R, if and only if $x \in Y$, the center of S. R will be regarded as a subring of $Q=Q(R)$, its complete ring of right quotients, and we will set $T=RC$, where C is the center of Q. The results of §§1 and 2 will be at our disposal.

THEOREM 7. *Either*

(*Case* 1) $$\phi(e_i) = c_i+f_i, \qquad i = 1, 2,$$

or

(*Case* 2) $$\phi(e_i) = c_i-f_i, \qquad i = 1, 2,$$

where $c_i \in C$, and f_1, f_2 are orthogonal idempotents of T whose sum is 1.

Proof. Set $\phi(e_i)=g_i$. Since $[[[xe_i]e_i]e_i]=[xe_i]$ for all $x \in S$ and ϕ is onto, it follows that $[[[yg_i]g_i]g_i]=[yg_i]$ for all $y \in R$. By Theorem 6 $g_i=c_i+f_i$, f_i an idempotent in $T=RC$, $c_i \in C$, $i=1, 2$. A direct calculation shows that

$$[[[[xe_1]e_2]e_1]e_2]+[[xe_1]e_2] = 0$$

for all $x \in S$. Since $c_i \in C$ we then have

$$[[[[yf_1]f_2]f_1]f_2]+[[yf_1]f_2] = 0$$

for all $y \in R$. Also, from $[e_1, e_2]=0$, we see that $[f_1, f_2]=0$. Lemma 9 then says that either $f_1f_2=0$ or $(1-f_1)(1-f_2)=0$. Therefore either $\phi(e_i)=c_i+f_i$, $\{f_i\}$ orthogonal idempotents, $i=1, 2$, or $\phi(e_i)=c_i+f_i=(c_i+1)-(1-f_i)=b_i-h_i$, $b_i \in C$, $\{h_i\}$ orthogonal idempotents, $i=1, 2$. Finally, from $0=[x, 1]=[x, e_1+e_2]$ for all $x \in S$, we see that $[y, f_1+f_2]=0$ for all $y \in R$. Therefore the idempotent f_1+f_2 lies in C and must be equal to 1.

Until further notice in this section we assume that Case 1 holds, i.e., $\phi(e_i)=c_i+f_i$, $i=1, 2$. We also recall the notation $S_{ij}=e_iSe_j$ and $T_{ij}=f_iTf_j$. Clearly, $S=\bigoplus \sum_{i,j=1}^{2} S_{ij}$, and, in view of Theorem 7, $T=\bigoplus \sum_{i,j=1}^{2} T_{ij}$.

LEMMA 11. *$\phi(S_{ij})\subseteq T_{ij}$, $i\neq j$.*

Proof. Let $x \in S_{12}$ and set $y=\phi(x)$. Since $x=[e_1[xe_2]]$ and $x=[e_1x]$ we have

$$\begin{aligned} y &= [f_1[yf_2]] = f_1yf_2+f_2yf_1 \\ &= f_1(f_1y-yf_1)f_2+f_2(f_1y-yf_1)f_1 = f_1yf_2-f_2yf_1. \end{aligned}$$

Therefore $y=f_1yf_2 \in T_{12}$, since char $T\neq 2$.

LEMMA 12. *$\phi(S_{ii})\subseteq T_{11}+T_{22}$.*

Proof. Let $x_1 \in S_{11}$ and set $\phi(x_1)=\sum_{i,j=1}^{2} y_{ij}=y$, $y_{ij} \in T_{ij}$. From $[x_1e_2]=0$ we conclude that $[yf_2]=y_{12}-y_{21}=0$. Hence $y_{12}=0=y_{21}$, i.e., $\phi(x_1) \in T_{11}+T_{22}$.

LEMMA 13. *$T_{ij}=\phi(S_{ij})C$, $i\neq j$.*

Proof. We first note that $T_{12}=f_1RCf_2=(f_1Rf_2)C$, and so we consider an element $f_1\phi(s)f_2c$, $c \in C$, $s \in S$. Writing $s=(s_{11}+s_{22})+s_{12}+s_{21}$, $s_{ij} \in S_{ij}$, we have

$$\phi(s) = \phi(s_{11}+s_{22})+\phi(s_{12})+\phi(s_{21}).$$

By Lemmas 11 and 12, this yields $f_1\phi(s)f_2c=f_1\phi(s_{12})f_2c=\phi(s_{12})c \in \phi(S_{12})C$.

Now consider the following sets:

$$\begin{aligned} A_1 &= \{a \in S_{11} \mid \phi(a) \in T_{11}+C\}, & A_2 &= \{a \in S_{11} \mid \phi(a) \in T_{22}+C\}, \\ B_1 &= \{b \in S_{22} \mid \phi(b) \in T_{11}+C\}, & B_2 &= \{b \in S_{22} \mid \phi(b) \in T_{22}+C\}. \end{aligned}$$

LEMMA 14. *$A_1=S_{11}$ or $A_2=S_{11}$, and $B_1=S_{22}$ or $B_2=S_{22}$.*

Proof. Suppose $A_1\neq S_{11}$ and $A_2\neq S_{11}$. Then there exists $x_1 \in S_{11}$ such that $x_1 \notin A_1$, i.e., $\phi(x_1)=u_1+u_2$, $u_1 \in T_{11}$, $u_2 \in T_{22}$, $u_2 \notin f_2C$. Since T is a prime ring, $I=T_{21}T_{12}$ is a nonzero ideal of the prime ring T_{22}. Furthermore, we see from Lemma 13 that $I=\phi(S_{21})\phi(S_{12})C$. By Lemma 10, the center of T_{22} is f_2C. Applying Lemma 6 to the ring T_{22} and the ideal I, we can find $w_2 \in I$ such that $u_2w_2-w_2u_2 \notin f_2C$. Since $I=\phi(S_{21})\phi(S_{12})C$ there exists $v_2=\phi(x_{21})\phi(x_{12}) \in I$ such that $u_2v_2-v_2u_2 \notin f_2C$. By Lemmas 11 and 12, we may write $v_2=\phi(a_1+a_2)$, where $a_i \in S_{ii}$. Set $y_1=x_1a_1-a_1x_1$. Then

$$\begin{aligned} \phi(y_1) &= \phi\{x_1(a_1+a_2)-(a_1+a_2)x_1\} = (u_1+u_2)v_2-v_2(u_1+u_2) \\ &= u_2v_2-v_2u_2 \notin f_2C. \end{aligned}$$

Now let $U=\{x \in S_{11} \mid \phi(x) \in T_{22}\}$. If $x \in U$ and $r \in S_{11}$, then

$$\phi(xr-rx) = \phi(x)\phi(r)-\phi(r)\phi(x) \in T_{22},$$

by Lemma 12. Therefore U is a Lie ideal of S_{11}. We have already shown that $y_1 \in U$. Since $\phi(y_1) \notin f_2C$, there exists $w_2 \in T_{22}$ such that $[\phi(y_1), w_2] \neq 0$. Because $w_2 \in T=RC$ we may write

$$w_2 = \sum_\lambda \phi(s^\lambda)c_\lambda = \sum_\lambda \{\phi(s_{11}^\lambda+s_{22}^\lambda)+\phi(s_{12}^\lambda)+\phi(s_{21}^\lambda)\}c_\lambda, \quad s_{ij}^\lambda \in S_{ij}, \quad c_\lambda \in C.$$

By Lemmas 11 and 12, it follows that $w_2=\sum_\lambda \phi(s_{11}^\lambda+s_{22}^\lambda)c_\lambda$, and so there exist $s_{11} \in S_{11}$, $s_{22} \in S_{22}$ such that $0 \neq [\phi(y_1), \phi(s_{11}s_{22})]=\phi(y_1s_{11}-s_{11}y_1)$. This shows that $y_1 \notin Z_1$, the center of S_{11}. Hence by Lemma 8, there is a nonzero ideal J of S_{11} for which $U \supseteq [J, S_{11}]$. Similarly $V=\{x \in S_{11} \mid \phi(x) \in T_{11}\}$ is a Lie ideal of S_{11} containing $[K, S_{11}]$ for some nonzero ideal K of S_{11}. $L=J \cap K$ is again a nonzero ideal of the prime ring S_{11}. Thus $U \supseteq [L, S_{11}]$ and $V \supseteq [L, S_{11}]$. Let $u \in U$ and $v \in V$. Then $\phi(uv-vu)=\phi(u)\phi(v)-\phi(v)\phi(u)=0$, i.e., $uv-vu=0$, or $[U, V]=0$. Thus $[[L, S_{11}], [L, S_{11}]]=0$ and in particular $[[L, L], [L, L]]=0$. Since L is itself a prime ring, L is commutative by Lemma 7. But it is well known that if a nonzero ideal of a prime ring is commutative then the ring itself is commutative. Therefore S_{11} is commutative, and we have a contradiction to the fact that $y_1=x_1a_1-a_1x_1 \neq 0$.

THEOREM 8. *Either*

(*Case* 1a) $\qquad A_1 = S_{11}$ *and* $B_2 = S_{22}$

or

(*Case* 1b) $\qquad A_2 = S_{11}$ *and* $B_1 = S_{22}$.

Proof. Suppose that neither of these two cases prevails. We may assume then, in view of Lemma 14, that $A_1=S_{11}$ and $B_1=S_{22}$. If $\phi(S_{11}) \subseteq f_1C+f_2C$, then $A_2=S_{11}$ and Case 1b would hold. If $\phi(S_{22}) \subseteq f_1C+f_2C$, then $B_2=S_{22}$ and Case 1a would hold. Therefore we may assume that there is an $x_1 \in S_{11}$ such that $\phi(x_1) =u_1+c_2$, $u_1 \in T_{11}$, $u_1 \notin f_1C$, $c_2 \in f_2C$, and that there is an $x_2 \in S_{22}$ such that $\phi(x_2)=v_1+d_2$, $v_1 \in T_{11}$, $v_1 \notin f_1C$, $d_2 \in f_2C$. As we have seen before, $\phi(S_{12})\phi(S_{21})C$ is a nonzero ideal of the prime ring T_{11}. Hence by Lemma 6 there exists

$$t_1 \in \phi(S_{12})\phi(S_{21})$$

such that $[[t_1u_1]u_1] \neq 0$. Since $t_1 \in R$, we may write, using Lemmas 11 and 12, $t_1=\phi(b_1+b_2)$, $b_i \in S_{ii}$. Setting $y_1=x_1b_1-b_1x_1 \in S_{11}$, we see that

$$\begin{aligned}\phi(y_1) &= \phi\{x_1(b_1+b_2)-(b_1+b_2)x_1\} = (u_1+c_2)t_1-t_1(u_1+c_2) \\ &= u_1t_1-t_1u_1 \notin f_1C.\end{aligned}$$

We claim that the additive subgroups U of T_{11} spanned by all elements of the form $\phi(s_1)c$, $s_1 \in S_{11}$, $\phi(s_1) \in T_{11}$, $c \in C$, is a Lie ideal of T_{11}. Indeed, if $t \in T_{11}$, we may write $t=\sum_\lambda \phi(w^\lambda)c_\lambda=\sum \phi(w_1^\lambda+w_2^\lambda)c_\lambda$, $c_\lambda \in C$, $w_i^\lambda \in S_{ii}$, making use of Lemmas 11 and 12. In view of this it suffices to note that

$$[\phi(x_1)c, \phi(w_1+w_2)d] = \phi[s_1, w_1](cd) \in U, \qquad c, d \in C.$$

Since $\phi(y_1) \in U$ there exists, by Lemma 8, an ideal I of T_{11} such that $U \supseteq [I, T_{11}]$. Similarly, the additive subgroup V of T_{11} spanned by all elements of the form $\phi(s_2)c$, $s_2 \in S_{22}$, $\phi(s_2) \in T_{11}$, $c \in C$, is a Lie ideal of T_{11} not contained in the center f_1C of T_{11}. So again by Lemma 8, $V \supseteq [J, T_{11}]$ for some nonzero ideal J of T_{11}. $K=I \cap J$ is a nonzero ideal of the prime ring T_{11}, and thus $U \supseteq [K, T_{11}]$ and $V \supseteq [K, T_{11}]$. But $[U, V]=0$ since $[\phi(s_1)c, \phi(s_2)d]=\phi[s_1, s_2](cd)=0$, $s_i \in S_{ii}$, $\phi(s_i) \in T_{ii}$, $c, d \in C$, $i=1, 2$. This in turn forces $[[K, K], [K, K]]=0$, a contradiction to Lemma 7.

For the remainder of this section we shall assume that Case 1a holds. Lemma 11 and Theorem 8 show us that

(A) if $x \in S_{ij}$, $i \neq j$, then $\phi(x)=x^* \in T_{ij}$,

(B) if $x \in S_{ii}$, then $\phi(x)=x^*+c$, $x^* \in T_{ii}$, $c \in C$.

We note that in (B) x^* and c are uniquely determined. Indeed, if $\phi(x)=x^*+c$ $=y+d$, $y \in T_{ii}$, $d \in C$, then $x^*-y \in C \cap T_{ii}=0$. Hence $y=x^*$ and $c=d$.

Relations (A) and (B) enable us to define in a natural way a mapping σ of S into T according to the rule $\sigma(x)=x^*$, $x \in S_{ij}$, $i, j=1, 2$. A mapping τ of S into C is then defined by $\tau(x)=\phi(x)-\sigma(x)$.

Lemma 15. *σ (and hence τ) is additive.*

Proof. It suffices to show that σ is additive on S_{ii}. Let x and y be elements of S_{ii}. Then

$$\begin{aligned}\sigma(x+y)-\sigma(x)-\sigma(y) &= \phi(x+y)-\tau(x+y)-\phi(x)+\tau(x)-\phi(y)+\tau(y)\\ &= \tau(x)+\tau(y)-\tau(x+y) \in C \cap T_{ii} = 0.\end{aligned}$$

Thus $\sigma(x+y)=\sigma(x)+\sigma(y)$.

Lemma 16. *Let $x \in S_{ii}$ and $y \in S_{ij}$, $i \neq j$. Then $\sigma(xy)=\sigma(x)\sigma(y)$.*

Proof. Let $x_1 \in S_{11}$ and let $x_{12} \in S_{12}$. Then

$$\begin{aligned}\sigma(x_1x_{12}) = \phi(x_1x_{12}) = \phi(x_1x_{12}-x_{12}x_1) &= \phi(x_1)\phi(x_{12})-\phi(x_{12})\phi(x_1)\\ &= \sigma(x_1)\sigma(x_{12})-\sigma(x_{12})\sigma(x_1) = \sigma(x_1)\sigma(x_{12}).\end{aligned}$$

Lemma 17. *Let $x \in S_{ij}$ and let $y \in S_{ji}$, $i \neq j$. Then $\sigma(xy)=\sigma(x)\sigma(y)$.*

Proof. Let $x_{12} \in S_{12}$, $a_{21} \in S_{21}$, and $s_{12} \in S_{12}$. We apply ϕ to $x_{12}a_{21}s_{12}+s_{12}a_{21}x_{12}$ $=[[x_{12}, a_{21}], s_{12}]$ and obtain

$$\phi(x_{12}a_{21}s_{12}+s_{12}a_{21}x_{12}) = \phi(x_{12})\phi(a_{21})\phi(s_{12})+\phi(s_{12})\phi(a_{21})\phi(x_{12}). \tag{1}$$

By Lemma 16, (1) becomes

$$\sigma(x_{12}a_{21})\sigma(s_{12})+\sigma(s_{12})\sigma(a_{21}x_{12}) = \sigma(x_{12})\sigma(a_{21})\sigma(s_{12})+\sigma(s_{12})\sigma(a_{21})\sigma(x_{12})$$

or

$$\{\sigma(x_{12}a_{21})-\sigma(x_{12})\sigma(a_{21})\}\sigma(s_{12}) = \sigma(s_{12})\{\sigma(a_{21})\sigma(x_{12})-\sigma(a_{21}x_{12})\}. \tag{2}$$

An analogous argument shows that, if $s_{21} \in S_{21}$, then

$$\sigma(s_{21})\{\sigma(x_{12}a_{21})-\sigma(x_{12})\sigma(a_{21})\} = \{\sigma(a_{21})\sigma(x_{12})-\sigma(a_{21})\sigma(x_{12})\}\sigma(s_{21}). \tag{3}$$

The subring U generated by T_{12} and T_{21} is an ideal of T and we know by Lemma 13 that $T_{ij}=\phi(S_{ij})C$, $i\neq j$. Therefore (2) and (3) say that the element

$$t = \sigma(x_{12}a_{21})-\sigma(x_{12})\sigma(a_{21})+\sigma(a_{21})\sigma(x_{12})-\sigma(a_{21}x_{12}) \tag{4}$$

commutes with every element of U. By Lemma 6 we have $t \in C$. Multiplication of (4) on the right by $\sigma(x_{12})$ yields

$$\sigma(x_{12}a_{21})\sigma(x_{12})-\sigma(x_{12})\sigma(a_{21})\sigma(x_{12}) = t\sigma(x_{12}). \tag{5}$$

By Lemma 16, (5) becomes

$$\sigma(x_{12}a_{21}x_{12})-\sigma(x_{12})\sigma(a_{21})\sigma(x_{12}) = t\sigma(x_{12}). \tag{6}$$

On the other hand, setting $s_{12}=x_{12}$ in (1) gives $\sigma(x_{12}a_{21}x_{12})=\sigma(x_{12})\sigma(a_{21})\sigma(x_{12})$. Therefore (6) becomes $0=t\sigma(x_{12})$, whence $t=0$. But then we have in particular from (4) that $\sigma(x_{12}a_{21})=\sigma(x_{12})\sigma(a_{21})$.

LEMMA 18. *Let $x, y \in S_{ii}$. Then $\sigma(xy)=\sigma(x)\sigma(y)$.*

Proof. Let $x_1, y_1 \in S_{11}$ and let $s_{12} \in S_{12}$. Then, using Lemma 16,

$$\sigma(x_1y_1)\sigma(s_{12}) = \sigma(x_1y_1s_{12}) = \sigma(x_1)\sigma(y_1s_{12}) = \sigma(x_1)\sigma(y_1)\sigma(s_{12}).$$

In particular, $\sigma(x_1y_1)-\sigma(x_1)\sigma(y_1)$ commutes with $\sigma(s_{12})$. Similarly, $\sigma(x_1y_1)-\sigma(x_1)\sigma(y_1)$ commutes with all the elements of $\phi(S_{21})$. Since $T_{ij}=\phi(S_{ij})C$, $i\neq j$, and the subring generated by T_{12} and T_{21} is an ideal of T, we see by Lemma 6 that $\sigma(x_1y_1)-\sigma(x_1)\sigma(y_1) \in C \cap T_{11}=0$. Thus $\sigma(x_1y_1)=\sigma(x_1)\sigma(y_1)$.

We come now to the main theorem of this paper for Case 1a.

THEOREM 9. *For Case* 1a, *σ is an isomorphism of S into T.*

Proof. From Lemmas 15, 16, 17, and 18, we know that σ is a homomorphism of S into T. Suppose for some $x=x_1+x_2+x_{12}+x_{21} \in S$ that $\sigma(x)=0$. From the definition of σ, $x_{12}=0=x_{21}$, and so $\sigma(x_1+x_2)=0$. From this $\sigma(x_1+x_2)\sigma(e_1)=\sigma(x_1)=0$, i.e., $\phi(x_1)=\tau(x_1) \in C$. Therefore $x_1 \in Y$, the center of S, which forces $x_1=0$. By the same token $x_2=0$, and thus we have shown that σ is a one-one mapping.

COROLLARY. *τ is an additive mapping of S into C such that $\tau(xy-yx)=0$ for all $x, y \in R$.*

5. **The main theorem: general case.** In this final section we continue with the assumptions made at the beginning of §3, but now superimpose until further notice the condition that Case 1b holds. Our immediate goal is thus to prove the analogue of Theorem 9. Lemma 11 and Theorem 8 combine to yield

(A) if $x \in S_{ij}$, $i \neq j$, then $\phi(x) = x^* \in S_{ij}$,

(B) if $x \in S_{11}$, then $\phi(x) = x^* + c$, $x^* \in T_{22}$, $c \in C$,

(C) if $x \in S_{22}$, then $\phi(x) = x^* + c$, $x^* \in T_{11}$, $c \in C$.

It is again clear that x^* and c are uniquely determined and that an additive mapping σ of S into T can be defined according to $\sigma(x) = x^*$, $x \in S_{ij}$, $i, j = 1, 2$. A mapping τ of S into C is then defined by $\tau(x) = \phi(x) - \sigma(x)$, $x \in S$.

LEMMA 19. *Let $x \in S_{ii}$ and let $y \in S_{ij}$, $i \neq j$. Then $\sigma(xy) = -\sigma(y)\sigma(x)$.*

Proof. Let $x_1 \in S_{11}$ and let $x_{12} \in S_{12}$. Then

$$\begin{aligned}\sigma(x_1 x_{12}) &= \phi(x_1 x_{12}) = \phi(x_1)\phi(x_{12}) - \phi(x_{12})\phi(x_1) \\ &= \sigma(x_1)\sigma(x_{12}) - \sigma(x_{12})\sigma(x_1) = -\sigma(x_{12})\sigma(x_1),\end{aligned}$$

since $\sigma(x_1) \in T_{22}$.

LEMMA 20. *Let $x \in S_{ij}$ and let $y \in S_{ji}$, $i \neq j$. Then $\sigma(xy) = -\sigma(y)\sigma(x)$.*

Proof. Let $x_{12} \in S_{12}$, $a_{21} \in S_{21}$, and $s_{12} \in S_{12}$. As in the proof of Lemma 17, we first obtain

$$\sigma(x_{12}a_{21}s_{12} + s_{12}a_{21}x_{12}) = \sigma(x_{12})\sigma(a_{21})\sigma(s_{12}) + \sigma(s_{12})\sigma(a_{21})\sigma(x_{12}). \tag{1}$$

By Lemma 19, (1) becomes

$$-\sigma(s_{12})\sigma(x_{12}a_{21}) - \sigma(a_{21}x_{12})\sigma(s_{12}) = \sigma(x_{12})\sigma(a_{21})\sigma(s_{12}) + \sigma(s_{12})\sigma(a_{21})\sigma(x_{12})$$

or

$$\sigma(s_{12})\{\sigma(a_{21})\sigma(x_{12}) + \sigma(x_{12}a_{21})\} = -\{\sigma(x_{12})\sigma(a_{21}) + \sigma(a_{21})\sigma(x_{12})\}\sigma(s_{12}). \tag{2}$$

A similar argument shows that, if $s_{21} \in S_{21}$, then

$$\sigma(s_{21})\{\sigma(x_{12})\sigma(a_{21}) + \sigma(a_{21}x_{12})\} = -\{\sigma(a_{21})\sigma(x_{12}) + \sigma(a_{12}a_{21})\}\sigma(s_{21}). \tag{3}$$

Continuing as in the proof of Lemma 17, we are able to conclude from (2) and (3) that

$$\sigma(x_{12}a_{21}) + \sigma(a_{21})\sigma(x_{12}) - \sigma(a_{21}x_{12}) - \sigma(x_{12})\sigma(a_{21})$$

is an element of C. From this, again as in the proof of Lemma 17, we finally obtain

$$\sigma(x_{12}a_{21}) = -\sigma(a_{21})\sigma(x_{12}).$$

We state without proof the analogue of Lemma 18.

LEMMA 21. *Let $x, y \in S_{ii}$. Then $\sigma(xy) = -\sigma(y)\sigma(x)$.*

THEOREM 10. *For Case* 1b, σ *is the negative of an anti-isomorphism of* S *into* T.

Proof. Lemmas 19, 20, and 21 combine to show that σ is the negative of a homomorphism ψ of S into T, where $\psi(x) = -\sigma(x)$, all $x \in S$. The same reasoning as in the proof of Theorem 9 is used to show that ψ is a one-one mapping of S into T.

COROLLARY. τ *is an additive mapping of* S *into* T *such that* $\tau(xy-yx)=0$ *for all* $x, y \in S$.

Having disposed of Case 1 completely, we assume now that Case 2 holds, that is, $\phi(e_i)=c_i-f_i$, $i=1, 2$ (see Theorem 7). Let θ: $x \to x'$ be an anti-isomorphism of R onto a ring R' (such exists). Then $-\theta$ is easily seen to be a Lie isomorphism of R onto R', whence $\psi = -\theta\phi$ is a Lie isomorphism of S onto R'. Let P be the complete ring of right quotients of R', let B be the center of P, and set $U=R'B$. Our aim is to show that Case 1 arguments apply to ψ, so that we may apply Theorems 9 and 10. In order to do this we require the following lemma.

LEMMA 22. θ *may be extended to an anti-isomorphism* χ *of* T *onto* U.

Proof. We first show that C is isomorphic to B. Let $c \in C$, and let

$$D = \{x \in R \mid cx \in R\}.$$

The image D' of D under θ is an ideal of R'. Define $g(d')=(cd)'$. Since $g(d'r') = g((rd)') = (c(rd))' = (r(cd))' = (cd)'r' = g(d')r'$, $r' \in R'$, we see that $g \in \mathrm{Hom}_{D'}(D', R')$. There exists a unique element $b \in P$ such that $bd' = g(d')$ for all $d' \in D'$. For $r' \in R'$, $d' \in D'$, we have $br'd' = b(dr)' = g(dr)' = (cdr)' = r'(cd)' = r'bd'$, showing that $br' = r'b$, i.e., $b \in B$. It can be verified that the mapping $c \to b$ is an isomorphism of C onto B. Next suppose $\sum_{i=1}^{n} r_i c_i = 0$, $r_i \in R$, $c_i \in C$. Let $D_i = \{x \in R \mid c_i x \in R\}$ and set $D = \bigcap D_i$. Then $D' = \bigcap D_i'$, and we see that, for $d' \in D'$, $(\sum r_i' b_i)d' = \sum r_i'(b_i d') = \sum r_i'(c_i d)' = (\sum c_i d r_i)' = (d \sum c_i r_i)' = 0$. Consequently $\sum r_i' b_i = 0$, which proves that the mapping χ: $\sum r_i c_i \to \sum r_i' b_i$ is well defined. It is then straightforward to verify that χ is an anti-isomorphism of $T=RC$ onto $U=R'B$.

Lemma 22 may now be applied in order to write $\psi(e_i) = -\theta(\phi(e_i)) = -\theta(c_i - f_i) = -\chi(c_i - f_i) = -\chi(c_i) + \chi(f_i) = b_i + g_i$, $i=1, 2$, $\{g_i\}$ orthogonal idempotents of U. This shows that Case 1 arguments apply to χ. By Theorems 9 and 10, therefore, $\psi = \sigma + \tau$, where σ is either an isomorphism or the negative of an anti-isomorphism of S into U and τ is an additive mapping of S into B which maps commutators into zero. Hence $-\chi\phi = -\theta\phi = \sigma + \tau$, from which we obtain $\phi = -\chi^{-1}\sigma - \chi^{-1}\tau = \sigma' + \tau'$, where σ' is either the negative of an anti-isomorphism or is an isomorphism of S into T and τ' is an additive mapping of S into C which maps commutators into zero.

From Theorems 9 and 10 and their corollaries, and from the discussion of the preceding paragraph, it is clear that we have completed the proof of the main theorem.

THEOREM 11. *Let S be a prime ring with* 1, *of characteristic different from* 2 *and* 3, *and containing two nonzero idempotents e_1 and e_2 whose sum is* 1. *Let ϕ be a Lie isomorphism of S onto a prime ring R with* 1. *Let Q be the complete ring of right quotients of R, let C be the center of Q, and let $T=RC$. Then ϕ is of the form $\sigma+\tau$, where σ is either an isomorphism or the negative of an anti-isomorphism of S into T and τ is an additive mapping of S into C which maps commutators into zero.*

It is only natural now to try to show that the results which we previously obtained for Lie isomorphisms in which the rings involved were either simple or primitive are actually corollaries of Theorem 11. Of course, since simple rings and primitive rings are necessarily prime rings, the obvious corollaries follow immediately. But sharper results are actually available, since in these special cases we now proceed to show that there is a close relationship between the ring $T=RC$ and the ring R. For simple rings, in fact, we prove that $T=R$.

LEMMA 23. *Let R be a simple ring with* 1, *with center Z, and let Q be the complete ring of right quotients of R, with center C. Then $Z=C$ (and hence $T=RC=R$).*

Proof. Let $c\in C$. The set $D=\{x\in R \mid cx\in R\}$, which we know is a dense right ideal of R, is clearly a two-sided ideal of R and hence equal to R. In particular, $c=c\cdot 1\in R$, and so $Z=C$.

Somewhat less obvious is the situation for primitive rings. Let R be a primitive ring with 1. We regard R as an irreducible ring of endomorphisms of an additive abelian group V. The set $\Delta=\mathrm{Hom}_R(V, V)$ is a division ring, and we let F denote the center of Δ. RF is a primitive ring with center F, and $R\subseteq RF$. On the other hand, if Q is the complete ring of right quotients of R and C is the center of Q, we may still regard R as a subring of $T=RC$.

THEOREM 12. *C is isomorphic to a subfield of F, and RC is isomorphic to a subring of RF.*

Proof. Let $c\in C$, and let $D=\{x\in R \mid cx\in R\}$. Choosing $v\neq 0\in V$, we may write $V=vD$, since D is an ideal of R. A mapping $f=f(c, v)$ of V into V is then defined by $vd\to v(cd)$ for all $d\in D$. Suppose $vd=0$. If $v(cd)\neq 0$, then $v(cd)D\neq 0$, and so there is an element $x\in D$ such that $v(cd)x\neq 0$. But $v(cd)x=vd(cx)=(vd)(cx)=0$, a contradiction. Therefore $v(cd)=0$, and f is well defined. For $d\in D$, $r\in R$, we see that $(vd)(rf)=\{v(dr)\}f=v(c(dr))=v(cd)r=(vd)fr$, i.e., $f\in\Delta$. Now let $d\in D$, $a\in\Delta$, and write $va=vx$ for some $x\in D$. Then $(vd)(af)=(va)df=(vx)df=v(xd)f$ $=v(cxd)=vx(cd)=(va)(cd)=v(cd)a=(vd)fa$. Therefore $af=fa$ and $f\in F$. We point out, too, that f is independent of the choice of v. Indeed, suppose $vd=v_1d_1$, but $v(dc)\neq v_1(d_1c)$, $v, v_1\in V$, $d, d_1\in D$. Then there is an $x\in D$ such that

$$\{v(dc)-v_1(d_1c)\}x \neq 0.$$

But $v(dc)x=(vd)(cx)=(v_1d_1)(cx)=v_1(d_1c)x$, a contradiction. We are now able to map C into F by $\rho: c\to f=f(v, c)=f(c)$. For $c_1, c_2\in C$, let $D_i=\{x\in R \mid c_ix\in R\}$,

$i=1, 2$, set $D=D_1D_2 \subseteq D_1 \cap D_2$, and let $v \neq 0 \in V$. For $d \in D$, we have $(vd)(f(c_1+c_2)$ $=((c_1+c_2)d)=v((c_1d)+(c_2d))=vdf(c_1)+vdf(c_2)$. This shows that ρ is additive. For $d \in D$, it is easy to see that $c_2d \in D_1$. Therefore, $(vd)f(c_1c_2)=v(c_1c_2)d=vc_1(c_2d)$ $=v(c_2d)f(c_1)=(vd)f(c_2)f(c_1)$. Hence ρ is multiplicative. Suppose $f(c_1)=0$. Then $V(D_1c_1)=0$, and so $D_1c_1=0$, which implies that $c_1=0$. We have thus shown that ρ is a ring isomorphism of C into F. We define a mapping γ of RC into RF according to

$$\sum_{i=1}^{n} r_ic_i \to \sum_{i=1}^{n} r_if_i$$

where $f_i=\rho(c_i)$. Suppose $\sum r_ic_i=0$, but $\sum r_if_i \neq 0$. Let $D_i=\{x \in R \mid c_ix \in R\}$, and set $D=\bigcap_{i=1}^{n} D_i$. For $v \in V$ and $d \in D$, we have

$$(vd)\left(\sum r_if_i\right) = \sum_i v(dr_i)f_i = \sum_i v(dr_i)c_i = v\left\{d\left(\sum r_ic_i\right)\right\} = 0.$$

Hence $\sum r_if_i=0$ and γ is well defined. It is clear that γ is a ring homomorphism, because ρ is a homomorphism. Finally, suppose $\sum r_if_i=0$. As above, let $D=\bigcap_{i=1}^{n} D_i$, let $v \in V$, and let $d \in D$. Then $0=vd(\sum r_if_i)=v\sum (dr_i)c_i=vd(\sum r_ic_i)$. Therefore $\sum r_ic_i=0$ and γ is one-one.

As a result of Theorem 11 and Theorem 12 we clearly have

THEOREM 13. *Let S be a prime ring with* 1, *of characteristic different from* 2 *and* 3, *and containing two nonzero orthogonal idempotents whose sum is* 1. *Let ϕ be a Lie isomorphism of S onto a primitive ring R with* 1. *Let R be regarded as an irreducible ring of endomorphisms of an additive abelian group V, let Δ be the division ring* $\mathrm{Hom}_R(V, V)$, *and let F be the center of Δ. Then ϕ is of the form $\sigma+\tau$, where σ is either an isomorphism or the negative of an anti-isomorphism of S into the primitive ring RF and τ is an additive mapping of S into F which maps commutators into zero.*

Except for the assumption that $1 \in R$, Theorem 13 is clearly a generalization of the main theorem of our first paper on Lie isomorphisms [2, p. 915, Theorem 4]. In that paper we assumed that S was primitive and contained three nonzero orthogonal idempotents whose sum was 1.

As a result of Theorem 11 and Lemma 23 (also as a corollary of Theorem 13), we have

THEOREM 14. *Let S be a prime ring with* 1, *of characteristic different from* 2 *and* 3, *and containing two nonzero orthogonal idempotents whose sum is* 1. *Let ϕ be a Lie isomorphism of S onto a simple ring R with center $Z \neq 0$. Then ϕ is of the form $\sigma+\tau$, where σ is either an isomorphism or the negative of an anti-isomorphism of S onto R* (*hence S must be simple*) *and τ is an additive mapping of S into Z which maps commutators into zero.*

Theorem 14 is the main theorem of our second paper on Lie isomorphisms [3].

Theorem 13 (and hence Theorem 11) has been illustrated with an example in [2, p. 916], which shows that the image of σ need not be contained in R.

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